ANALYTIC SOLUTION FOR THE FORCED MEAN CROSS-SHORE FLOW IN THE SURF ZONE

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Analytical solutions to the forced horizontal momentum equations are found for the local vertical structure of mean flow driven by surface wave breaking within a saturated surf zone. Similar to the theoretical development of flows within the wave bottom boundary layer driven by arbitrary free stream wave motions (Foster, et al., 1999), the oscillating surface boundary condition is distributed through the water column by transforming the vertical coordinate, $z' = z + h/\eta + h$ where h is the still water depth and η is the fluctuating sea surface elevation. The transformation leads to local analytical solutions of the vertical structure of time-averaged currents driven by surface wave breaking without explicitly defining wave trough levels or surface mass fluxes. The solutions have the attractive attribute that they estimate the sub-surface flow contributed from surface forcing by wave breaking. The total mean flow includes these solutions plus contributions by the Stokes Drift, not explicitly considered herein.

INTRODUCTION

Observations in both the field (*e.g.*, Smith, et al., 1992; Haines and Sallenger, 1994; Masselink and Black, 1995; Garcez Faria, et al., 2000; and others) and the laboratory (*e.g.*, Stive and Wind, 1986; Ting and Kirby, 1994; and others) universally show strong vertical variation in the cross-shore mean flow. The seaward-directed mean cross-shore current, generally referred to as the undertow, is in general well sampled in these experiments. The driving force for the undertow has long been linked to the vertical imbalance in radiation stresses and the mean pressure (setup) gradient (*e.g.*, Dyhr-Nielsen and Sorensen, 1970; Svendsen, 1984; Stive and Wind, 1986; Deigaard, et al., 1991; and many others) with a number of numerical models that provide a reasonable representation of the seaward component of the undertow.

In general, these studies restrict the analysis to the sub-trough level where there is always water and any Eulerian measurements are easily compared with model predictions. In the region between the wave trough and crest, any particular elevation in the vertical is alternatively covered and uncovered by water, with increasingly more out-of-water time the closer to the wave crest. The mean flow between the crest and trough when there is water present is always onshore (except, perhaps, in the present of strong rip currents which will not be

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considered herein), creating a shoreward mass flux that balances the return flow below the trough. This Stokes Drift (Phillips, 1980) is considered in models (*e.g.*, Walstra, et al., 2000), but by definition, the Stokes Drift only considers the contribution by non breaking wave motions. Numerical models have been developed that include the effects of surface mass flux contributed by both the Stokes Drift and by momentum fluxes associated with wave breaking in the surf zone (*e.g.*, Stive and Wind, 1982; Garcez Faria, et al., 2000). However, the contribution above the wave trough level is not specified in detail, and thus the vertical variation of the mean flow between the trough and crest is not estimated (considering only periods when water is present at any given elevation).

In this work, we derive analytic solutions for the vertical variation in time-averaged mean flow from the sea bed to the level of the wave crest by transforming the forced two-dimensional momentum equations following Foster, et al. (1999) who considered the vertical structure of the flow in the wave bottom boundary layer forced by an oscillating free stream velocity. The solutions have hyperbolic form, and depend on both the (unspecified) surface forcing by breaking and a vertically uniform eddy viscosity. Above the trough level, the solution describes the time-averaged flow forced by surface breaking only for the period of time when water is present.

THEORY

The forced horizontal momentum equation for the two-dimensional cross-shore flow can be written (using continuity and assuming no alongshore flow) as

$$\frac{\partial}{\partial x}\left(P + \rho\hat{u}^2\right) + \frac{\partial}{\partial z}\rho\hat{u}\hat{w} = 0 \tag{1}$$

(Stive and Wind, 1982), where the pressure $P = \rho g(\eta - z) - \rho\hat{w}^2$ with η the sea surface elevation, $\hat{u}$ and $\hat{w}$ are the cross-shore and vertical velocities, ρ is density, and x and z are the horizontal and vertical Cartesian coordinates with z positive upward from the still water level. The velocities are assumed to be composed of mean (U), wave ($\tilde{u}$, $\tilde{w}$), and turbulent (u', w') components, such that

$$\hat{u} = U + \tilde{u} + u' \tag{2}$$

$$\hat{w} = \tilde{w} + w' \tag{3}$$

Time-averaging (1) thus leads to a governing equation for the mean flow field, that unfortunately cannot be solved analytically for the vertical structure because the surface boundary conditions (at $z = \eta$) are functions of space and time,

$$\hat{u}(\eta) = \hat{u}(x,\, t) \tag{4}$$

$$\hat{w}(\eta) = \hat{w}(x,\, t) \tag{5}$$

In previous work, the vertical flow structure was partitioned into two layers, above and below the wave trough level, and mass conserved over the vertical such that the onshore mass flux above the trough level is balanced by an imposed depth uniform mean return flow, U_r, below the trough (e.g., Garcez Faria, et al., 2000). This methodology leads to an equation describing only the mean flow below the wave trough, given by

$$\frac{\partial}{\partial z}\left(\rho\nu\frac{\partial U}{\partial z}\right)=\frac{1}{2}\frac{\partial}{\partial x}\left[\rho\overline{\left(\tilde{u}^2-\tilde{w}^2\right)}\right]+\rho g\frac{\partial\overline{\eta}}{\partial x}+\frac{\partial\rho U_r^2}{\partial x} \tag{6}$$

where the over-bar indicates time averaging over the wave period. In (6) it has been assumed that the time-averaged turbulent shear stresses are determined by

$$-\rho\overline{u'w'}=\rho\nu\frac{\partial U}{\partial z} \tag{7}$$

with ν an eddy viscosity. Numerical solutions for the mean flow structure below the trough level are determined by the form of ν, and by various methods for specifying U_r from the vertical mass balance. No form for the structure of the mean flow above the trough level is found, limiting the solution to numerically calculated sub-trough currents.

Following Foster, et al. (1999), we show that an analytical solution can be found for the forced cross-shore flow spanning the water column from the bottom to the fluctuating sea surface ($-h<z<\eta$) in the presence of surface wave breaking. In the oscillatory wave bottom boundary layer model of Foster, et al. (1999) the vertical coordinate is transformed so that boundary conditions at the free stream and at the bottom are constant (and zero).

The coordinate transformation takes the form

$$z'=\frac{z+h}{\eta+h} \tag{8}$$

so that the velocities in the transformed system are given by

$$u(z')=\hat{u}-z'\hat{u}_o \tag{9}$$

$$w(z')=\hat{w}-z'\hat{w}_o \tag{10}$$

where we have used the simplified notation

$$\hat{u}(\eta)=\hat{u}_o \tag{11}$$

$$\hat{w}(\eta)=\hat{w}_o \tag{12}$$

The boundary conditions become zero both at the surface ($z'=1$) and at the bottom ($z'=0$).

$$u(0) = u(1) = 0 \tag{13}$$

$$w(0) = w(1) = 0 \tag{14}$$

Inserting (8)-(12) into (1), using (2)-(3), assuming shallow water so that the approximations $\tilde{w} = \tilde{w}_o z'$ and $\tilde{u} = \tilde{u}_o$ can be made, and the transformed time-averaged governing equation becomes

$$\frac{\partial}{\partial x} g\overline{\eta} - \overline{\frac{u_o w_o}{\eta + h}} - \overline{\frac{2w_o}{\eta + h}} U(z') = \overline{\frac{\nu}{(\eta + h)^2}} \frac{\partial^2}{\partial z'^2} U(z') \tag{15}$$

Equation (15) has form

$$\left(A(x) - \frac{\partial^2}{\partial z'^2} \right) U(z') - B(x) = 0 \tag{16}$$

with hyperbolic solutions, where A and B are functions only of x. Using the surface and bottom boundary conditions, solution to (15) is given by

$$U(z') = H \left[1 - \frac{\sinh\left(\sqrt{G} z'\right)}{\sinh \sqrt{G}} - \frac{\sinh\left(\sqrt{G}\,(1 - z')\right)}{\sinh \sqrt{G}} \right] \tag{17}$$

where G and H are functions of the surface wave field independent of depth

$$G(x) = \overline{\frac{2w_o}{\eta + h}} \Big/ \overline{\frac{\nu}{(\eta + h)^2}} \tag{18}$$

$$H(x) = \left(\frac{\partial}{\partial x} g\overline{\eta} + \overline{\frac{u_o w_o}{\eta + h}} \right) \Big/ \overline{\frac{w_o}{\eta + h}} \tag{19}$$

Transformation of (17) back to the original, untransformed coordinate system yields

$$U(z) = \overline{\frac{z + h}{\eta + h}} U_o + H \left[1 - \frac{\sinh\left(\sqrt{G}\,\overline{\frac{z + h}{\eta + h}}\right)}{\sinh \sqrt{G}} - \frac{\sinh\left(\sqrt{G}\left(1 - \overline{\frac{z + h}{\eta + h}}\right)\right)}{\sinh \sqrt{G}} \right] \tag{20}$$

Assuming mass conservation over the vertical in the cross-shore direction,

$$\int_{-h}^{a} U(z) dz = 0 \tag{21}$$

an estimate of the mean flow at the surface, U_o, can be found in terms of G and H,

$$U_o = 2H\left[\frac{1}{\overline{\dfrac{a+h}{\eta+h}}} - \frac{\cosh\left(\sqrt{G}\,\overline{\dfrac{a+h}{\eta+h}}\right) - 1 - \cosh\left(\sqrt{G}\left(1-\overline{\dfrac{a+h}{\eta+h}}\right)\right) + \cosh\sqrt{G}}{\left(\overline{\dfrac{a+h}{\eta+h}}\right)^2 \sqrt{G}\sinh\sqrt{G}}\right] \quad (22)$$

RESULTS

Solutions given by (20) and (22) are shown in Figure 1 for a saturated surf zone with wave amplitude to water depth ratio $a/h = 0.17$ and with the arbitrarily chosen values $G = -0.3$ and $H = 10\, m/s$. Sea surface elevation is given by a single linear wave with arbitrary frequency. The solutions are as expected, with surface flow onshore above the trough with maximum value at the elevation of the wave crest, and subsurface flow (*i.e.*, the undertow) with maximum below the wave trough level.

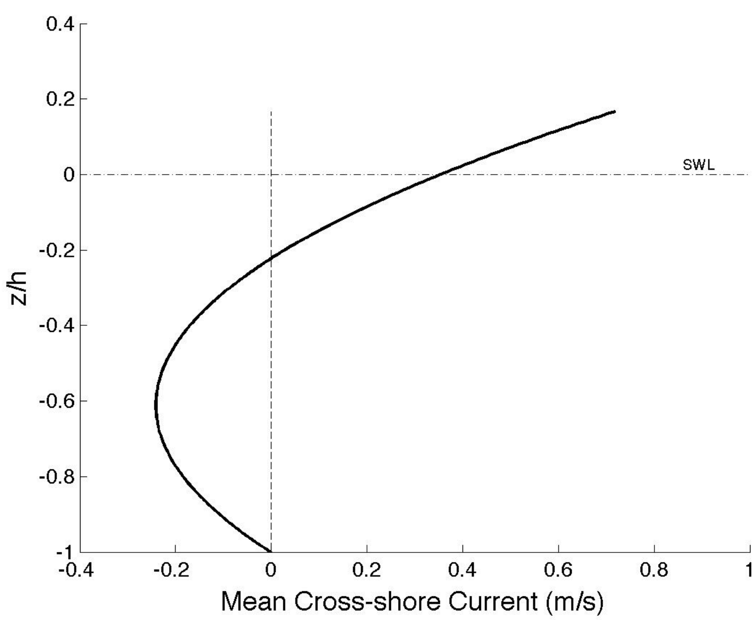

Figure 1. Example analytical solution for the vertical structure of the forced mean cross-shore flow given by equations (20) and (22) with *G* = -0.3*,* *H* = 10 *m/s,* and *a/h* = 0.17.

Note that in the solution the flow field extends upward beyond the still water level to a position equivalent to the wave amplitude, $z = a$. The water does not occupy the position above the trough (at elevations $z > -a$) at all times and only reaches the crest of the waves at the moment the breaking wave passes. The velocity at the crest of the mean flow profile is maximum, only occurring during the forced onshore flows right at the crest of the breaking wave. At lower elevation, but above the trough level, the time-averaged flow (occurring only when water is present at that elevation) is lower and smoothly connects with the sub-trough flow.

The form for $U(z)$ is determined by the relative magnitudes of G and H. With H held constant (e.g., a particular forcing condition independent of ν), and then choosing a range of G (which varies inversely with ν) allows an examination of the flow structure in response to changes in the eddy viscosity. Figure 2 shows $U(z)$ for a range of G and with H held constant. As the eddy viscosity increases, the vertical mixing is stronger and the vertical variation in mean flow is reduced (i.e., more uniform throughout the water column), as expected. By holding G constant and varying H, the impact of stronger forcing is qualitatively examined. Figure 3 show $U(z)$ for a range of H with the mixing (represented by G) constant. For higher wave forcing both the onshore surface flow and subsurface return flows are more pronounced.

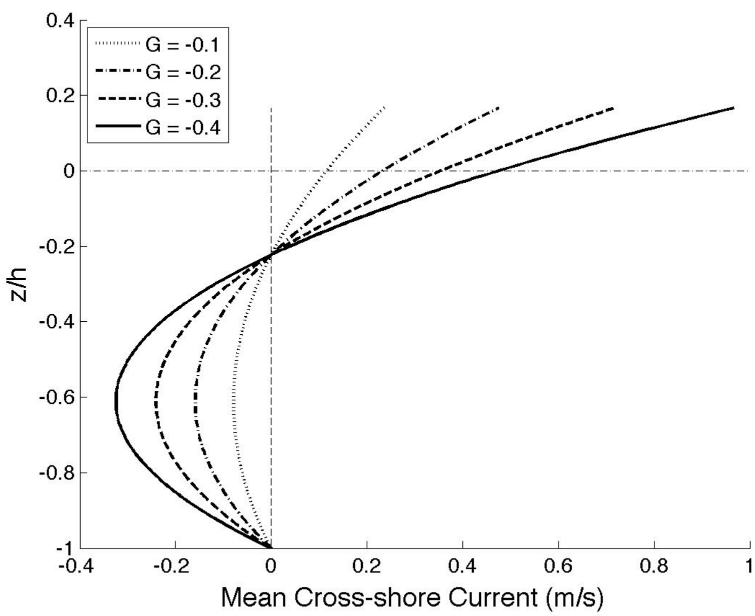

Figure 2. Analytical solutions for the forced mean cross-shore flow with *H* = 10 *m/s* and over a range of *G*, showing the effect of vertical mixing through variation in eddy viscosity.

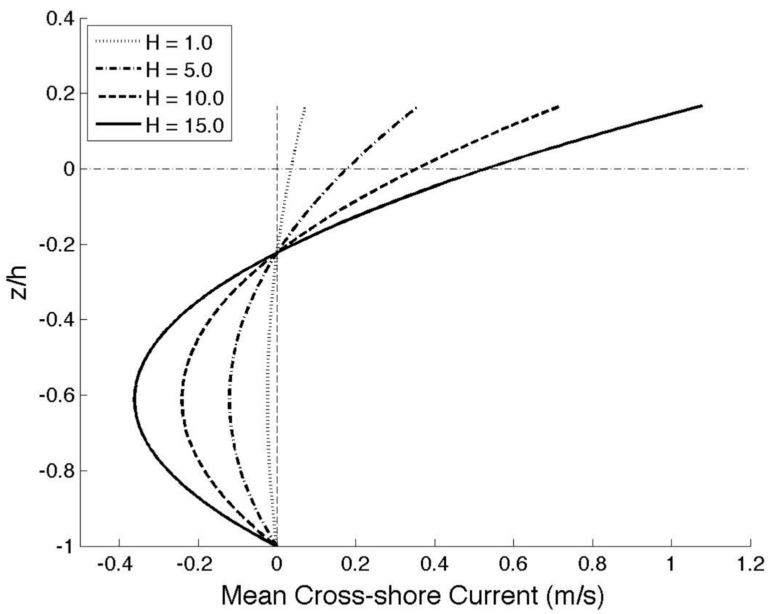

Figure 3. Analytical solutions for the forced mean cross-shore flow with *G* = -0.3 and over a range of *H*, showing the effect of forcing magnitude on the vertical variation in flow.

DISCUSSION

The variable G qualitatively determines the effect eddy mixing has on the vertically varying mean flow field. The form for G was found analytically only because the equations were grossly simplified by letting the eddy viscosity be uniform over depth. In nature, we would expect ν to vary over the water column and be strongly dependent on the turbulent structure both near the surface where wave breaking occurs and the wave-current bottom boundary layer.

The magnitude of the forcing by breaking processes (including setup and surface shear stresses induced by breaking) is described qualitatively by H, independent of eddy viscosity. When H goes to zero, the wave breaking is turned off, no external forcing occurs, and the forced solution goes to zero. This does not mean that the time-averaged flow at a given elevation above the trough is zero in the absence of wave breaking. Contributions to the mean cross-shore flow structure occur in the presence of waves (non breaking or breaking) due to the Stokes Drift, not accounted for herein.

Solutions were found under the assumption of shallow linear waves so that the horizontal wave velocities were uniform over depth and thus represented by surface values, and similarly so that the vertical wave velocities varied linearly over depth and could also be represented by surface values. This simplification is necessary for analytic solutions, but greatly over simplifies the complexities in natural breaking waves with strong nonlinearities. Thus, the solutions are only useful in a qualitative manner, but appear to represent the gross, expected form of the mean cross-shore flow. Consideration of higher order wave interactions could make the results applicable to field situations but would require numerical solutions.

When surface waves pass a given location in the surf zone, the elevation of the surface moves up and down. Thus an observing system that follows the water surface measures the mean and oscillatory surface flow field that depends on the elevation, or phase, at which the measurement is made. A time-average over the wave period is equivalent to a time average over the range of elevations spanned, that is, from the wave trough to the wave surface. Hence, an observing system following the surface flow can be time averaged and compared qualitatively with the solutions given by (20) and (22) provided an account of the Stokes Drift is made.

CONCLUSIONS

Analytical solutions to the forced horizontal momentum equations are found for the local vertical structure of mean flow within a saturated surf zone. Similar to the theoretical development of flows within the wave bottom boundary layer driven by arbitrary free stream wave motions (Foster, et al., 1999), the oscillating surface boundary condition is distributed through the water column allowing for local analytical solutions of the vertical structure of time-averaged currents to be found without explicitly defining wave trough levels or surface mass fluxes.

The forced horizontal momentum equations for the two-dimensional cross-shore flow are given by Stive and Wind (1982). Assuming the velocities to be composed of mean, wave, and turbulent components and time-averaging leads to a governing equation for the mean flow field; however, the surface boundary conditions are functions of space and time precluding direct analytic solutions for the mean flow structure (Garcez Faria, et al., 2000). In this work, we show that a complete analytical solution spanning the water column from the bottom to the surface can be found. Following Foster, et al. (1999), the vertical coordinate is transformed by $z' = z + h / \eta + h$, so that the boundary conditions become zero both at the surface ($z = \eta$) and at the bottom ($z = -h$). After significant simplification, and assuming mass conservation over the vertical, the transformed time-averaged governing equation can be shown to have hyperbolic solutions.

The solutions are as expected, with surface flow onshore above the trough with maximum value at the elevation of the wave crest, and subsurface flow (*i.e.* the undertow) with maximum below the wave trough level. The form for the undertow profile is determined by the relative magnitudes of the eddy mixing and the wave forcing. As the eddy viscosity increases, the vertical mixing is stronger and the vertical variation in mean flow is reduced, also as expected. For higher wave forcing, both the onshore surface flow and subsurface return flow are stronger. Total flow includes both the forced solution presented here that includes the effects of surface wave breaking, and from contributions due to the Stokes Drift, not presently considered.

ACKNOWLEDGMENTS

This research was sponsored by the Office of Naval Research, Coastal Geosciences program. The work was done while the author was at the Byrd Polar Research Center, Ohio State University.

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